Untwisting Noncommutative $\mathbb{R}^d$ and the Equivalence of Quantum Field Theories

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Abstract

We show that there is a duality exchanging noncommutativity and non-trivial statistics for quantum field theory on $\mathbb{R}^d$. Employing methods of quantum groups, we observe that ordinary and noncommutative $\mathbb{R}^d$ are related by twisting. We extend the twist to an equivalence for quantum field theory using the framework of braided quantum field theory. The twist exchanges both commutativity with noncommutativity and ordinary with non-trivial statistics. The same holds for the noncommutative torus.

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Introduction

Quantum field theory with noncommuting coordinates was proposed a long time ago with the hope to regularise divergencies [1]. A more ambitious motivation comes from the possible role of noncommutative geometry in the ongoing struggle to unite gravity with quantum field theory. Needless to say, the issue of even defining a quantum field theory on a noncommutative space is highly nontrivial. However, a generalisation of quantum field theory to noncommutative spaces with symmetries has recently been proposed [2].

In this paper we consider coordinate commutation relations of the form

$$[x^\mu, x^\nu] = i\,\theta^{\mu\nu}$$

in d dimensions, where θ is a real-valued antisymmetric matrix. This can also be viewed as equipping the algebra of functions on $\mathbb{R}^d$ with a deformation quantised multiplication known as a Moyal $\star$-product [3]. We refer to this space-time algebra as "noncommutative $\mathbb{R}^d$". We shall also consider the toroidal compactification known as the "noncommutative torus". It served as an early example of a noncommutative geometry for Connes [4].

Commutation relations of the type considered here were proposed by Doplicher, Fredenhagen and Roberts based on an analysis of the constraints posed by general relativity and Heisenberg's uncertainty principle [5]. (For other approaches at noncommutative space-times see [6, 7, 8, 9].) They also initiated the study of quantum field theory on this kind of space-time. (For an alternative approach to quantum field theory with generalised uncertainty relations see [10]). Basic results for Feynman diagrams relating the noncommutative and the commutative setting were obtained by Filk [11]. With the emergence of the noncommutative torus in string theory [12], quantum field theory on such a space has received a much wider interest, see [13] and references therein. Recently, the perturbation theory has been of particular interest with the investigation of divergencies and renormalisability, see e.g. [14, 15].

Our treatment makes essential use of the theory of quantum groups and braided spaces (see e.g. [16]). The starting point is the observation by Watts [17] that ordinary and noncommutative $\mathbb{R}^d$ are related by a certain 2-cocycle. This cocycle is associated with the translation group (which we also denote by $\mathbb{R}^d$) and induces a "twist". While the twist turns $\mathbb{R}^d$ into itself as a group, it turns $\mathbb{R}^d$ into noncommutative $\mathbb{R}^d$ as a representation. Importantly, the concepts of cocycle and twist used here are dual to those of ordinary group cohomology and arise only from the quantum group point of view. Employing the framework of braided quantum field theory [2] enables us to describe quantum field theory on both commutative and noncommutative $\mathbb{R}^d$ in a purely algebraic language. This allows the extension of the twist relating the two spaces to an equivalence between the quantum field theories

living on them. Underlying is an equivalence of categories of representations. However, the noncommutative $\mathbb{R}^d$ in this context carries a non-trivial statistics.

The noncommutative $\mathbb{R}^d$ with ordinary statistics (which is the space considered in the literature) on the other hand is related by the same twist to commutative $\mathbb{R}^d$ with non-trivial statistics. Here as well, we obtain an equivalence of quantum field theories on the two spaces. In this case it is really a duality exchanging noncommutativity and non-trivial statistics. In terms of perturbation theory, the duality exchanges a setting where vertices are noncommutative with a setting where vertices are commutative, but crossings carry an extra Feynman rule. As a byproduct, Filk's results are an immediate consequence. Finally, we investigate further space-time symmetries and gauge symmetry. We find that while they are preserved by the twist (as quantum group symmetries) they are broken by removing the non-trivial statistics from noncommutative $\mathbb{R}^d$. Although the discussion is in terms of $\mathbb{R}^d$ for convenience, it applies identically to the torus (except for the extra space-time symmetries).

Our equivalence result also suggests that a noncommutativity of the kind considered here really is to "weak" to be able to regularise a quantum field theory. What one needs for that purpose is a "stronger" noncommutativity in the form of a strict braiding (with a double-exchange not being the identity). This is for example provided by q-deformations of Lie groups. That quantum field theory can indeed be regularised in this way was demonstrated in [2].

Note that the concept of twisting has been used to relate quantum space-times [18] and quantum mechanical models [19] before. Also, 2-cocycles of ordinary group cohomology have been used to obtain noncommutative spaces in the context of matrix theory [20].

The article starts in Section 1 with the mathematical basis concerning twisting, twist equivalence, and the relation with deformation quantisation. This is mainly to equip the reader with the structures, formulas, and statements required later and contains only minimal new results. Section 2 looks at noncommutative $\mathbb{R}^d$ from the quantum group point of view and establishes the equivalence with ordinary $\mathbb{R}^d$ via twisting. The torus is treated as a special case. The main part of the article is section 3, where quantum field theory on noncommutative $\mathbb{R}^d$ is analysed. The twist is extended to quantum field theory, leading to the equivalences mentioned above. Perturbative consequences are investigated. Space-time and gauge symmetry are considered at the end.

The reader less familiar with the theory of quantum groups is encouraged to start by reading Section 3 and then return to Sections 1 and 2 for the foundations.

1 Foundations: Twists and Equivalence

This section provides the necessary mathematical foundations in the form needed for our treatment. It is a review of known material except perhaps for the two rather trivial Lemmas. A useful standard reference for the general theory of quantum groups and braided spaces is Majid's book [16]. We use the notations $\Delta, \epsilon, \mathrm{S}$ for coproduct, counit and antipode of a Hopf algebra respectively. We use Sweedler's notation $\Delta\, a = a_{(1)} \otimes a_{(2)}$ for coproducts and a similar notation $v \mapsto v_{(1)} \otimes v_{(2)}$ for left coactions. $\Bbbk$ denotes a general field.

We recall that a coquasitriangular structure $\mathcal{R} : H \otimes H \to \Bbbk$ on a Hopf algebra H provides a braiding on its category of comodules. More precisely, given any two comodules V and W there is an intertwining map $\psi : V \otimes W \to W \otimes V$ (the "braiding"). ψ can be seen as a replacement of the concept of an ordinary transposition which would be an intertwiner for representations of an ordinary group. For left comodules, ψ is given in terms of $\mathcal{R}$ as

$$\psi(v \otimes w) = \mathcal{R}(w_{(1)} \otimes v_{(1)})\, w_{(2)} \otimes v_{(2)}. \tag{1}$$

If $\mathcal{R}$ satisfies an extra condition, then for H-invariant elements the braiding ψ is just the same as the flip map:

Lemma 1.1. *Let H be a Hopf algebra with coquasitriangular structure $\mathcal{R} :$ $H \otimes H \to \Bbbk$ satisfying the property $\mathcal{R}(\mathrm{S}\, a_{(1)} \otimes a_{(2)}) = \epsilon(a)$. Then for left comodules V and W and $v \otimes w \in V \otimes W$ H-invariant we have $\psi(v \otimes w) = w \otimes v$.*

Proof. $w_{(1)} \otimes v_{(1)} \otimes w_{(2)} \otimes v_{(2)} = \mathrm{S}\, v_{(1)} \otimes v_{(2)} \otimes w \otimes v_{(3)}$ due to invariance. Inserting this into (1) gives the desired result. $\qquad\square$

We note that this property extends to cyclic permutations of invariant elements in multiple tensor products (just replace V or W by a multiple tensor product). This fact will be of interest later.

We turn to the concept of a twist of a Hopf algebra. Let H be a Hopf algebra and $\chi : H \otimes H \to \Bbbk$ be a unital 2-cocycle, i.e., a linear map that has a convolution inverse and satisfies the properties

$$\chi(a_{(1)} \otimes b_{(1)})\, \chi(a_{(2)} b_{(2)} \otimes c) = \chi(b_{(1)} \otimes c_{(1)})\, \chi(a \otimes b_{(2)} c_{(2)}) \tag{2}$$

$$\text{and} \quad \chi(a \otimes 1) = \chi(1 \otimes a) = \epsilon(a). \tag{3}$$

This defines a twisted Hopf algebra H' with the same coalgebra structure and unit as H. Its product and antipode are given by

$$a \bullet b = \chi(a_{(1)} \otimes b_{(1)})\, a_{(2)} b_{(2)}\, \chi^{-1}(a_{(3)} \otimes b_{(3)}), \tag{4}$$

$$\mathrm{S}'\, a = U(a_{(1)})\, \mathrm{S}\, a_{(2)} U^{-1}(a_{(3)}) \quad \text{with} \quad U(a) = \chi(a_{(1)} \otimes \mathrm{S}\, a_{(2)}).$$

If H carries a coquasitriangular structure $\mathcal{R} : H \otimes H \to \Bbbk$, then H' carries an induced coquasitriangular structure $\mathcal{R}'$ given by

$$\mathcal{R}'(a \otimes b) = \chi(b_{(1)} \otimes a_{(1)}) \, \mathcal{R}(a_{(2)} \otimes b_{(2)}) \, \chi^{-1}(a_{(3)} \otimes b_{(3)}). \tag{5}$$

The main result we need in the following is that the twist that turns H into H' extends to the corresponding comodule categories and establishes an equivalence. We formulate it here for left comodules. It is ultimately due to Drinfeld. See [21] for a dual version in the setting of quasi-Hopf algebras.

Theorem 1.2 (Drinfeld). *Let H be a Hopf algebra, $\chi : H \otimes H \to \Bbbk$ a unital 2-cocycle, H' the corresponding twisted Hopf algebra. There is an equivalence of monoidal categories $\mathcal{F} : {}^H\mathcal{M} \to {}^{H'}\mathcal{M}$. $\mathcal{F}$ leaves the coaction unchanged. The monoidal structure is provided by the natural equivalence*

$$\sigma : \mathcal{F}(V) \otimes \mathcal{F}(W) \to \mathcal{F}(V \otimes W)$$
$$v \otimes w \mapsto \chi(v_{(1)} \otimes w_{(1)}) \, v_{(2)} \otimes w_{(2)}$$

for all $V, W \in {}^H\mathcal{M}$. If H is coquasitriangular, the equivalence $\mathcal{F}$ becomes a braided equivalence.

Let us remark that an H-invariant element of a 2-fold tensor product remains the same under twist if χ satisfies an extra property. This is the following Lemma.

Lemma 1.3. *In the context of Theorem 1.2 let V and W be H-comodules and $v \otimes w \in V \otimes W$ be H-invariant. Assume further that χ satisfies $\chi(a_{(1)} \otimes \mathrm{S}\, a_{(2)}) = \epsilon(a)$. Then $\sigma^{-1}(v \otimes w) = v \otimes w$.*

Proof. First observe that the mentioned property of χ is automatically satisfied by χ^{-1} as well. Then use invariance in the form $v_{(1)} \otimes w_{(1)} \otimes v_{(2)} \otimes w_{(2)} = v_{(1)} \otimes \mathrm{S}\, v_{(2)} \otimes v_{(3)} \otimes w$ and apply σ^{-1}. $\qquad\square$

The equivalence σ extends to multiple tensor products by associativity. We denote the extension to an n-fold tensor product by σ_n. The induced transformation of a morphism α sending an n-fold to an m-fold tensor is $\sigma_m^{-1} \circ \alpha \circ \sigma_n$. In particular, we can apply this to the product map of an algebra (see [16] for similar examples).

Example 1.4. *Let H be a Hopf algebra, A a left H-comodule algebra and χ a unital 2-cocycle over H. Then A' built on A with the new multiplication*

$$a \star b = \chi(a_{(1)} \otimes b_{(1)}) \, a_{(2)} b_{(2)}$$

is an H'-comodule algebra. Note that the associativity of the product is ensured by the cocycle condition (2).

It is well known that for a Lie group G, a twist of its Hopf algebra of functions provides a (strict) deformation quantisation. More interestingly in our context, a twist also provides a deformation quantisation on any manifold M that G acts on. In particular, taking $M = G$ leads to a different (non-strict) deformation quantisation. Recall that a deformation quantisation of a manifold M is an associative linear map $\star : \mathcal{C}(M) \otimes \mathcal{C}(M) \to \mathcal{C}(M)[[\hbar]]$ which satisfies $f \star g = fg + \mathcal{O}(\hbar)$ and $f \star g - g \star f = \hbar\{f, g\} + \mathcal{O}(\hbar^2)$ where $\{\cdot, \cdot\}$ is a Poisson bracket on M. One usually also requires that the $\star$-product is defined for all orders in $\hbar$ by bidifferential operators (see e.g. [22]). The following Proposition (in dual form) is due to Drinfeld [23].

Proposition 1.5 (Drinfeld). *Let G be a Lie group acting on a manifold M. Denote by $H = \mathcal{C}(G)$ the (topological) Hopf algebra of functions on G and by $A = \mathcal{C}(M)$ the H-comodule algebra of functions on M. Then a unital 2-cocycle $\chi : H \otimes H \to \mathbb{C}[[\hbar]]$ so that $\chi(f \otimes h) = \epsilon(f)\,\epsilon(h) + \mathcal{O}(\hbar)$ defines a deformation quantisation on A.*

Finally, let us mention that for commutative Hopf algebras C and H with C a left H-comodule algebra and coalgebra there is a commutative semidirect product Hopf algebra $C \rtimes H$. It is freely generated by C and H as a commutative algebra. Its coproduct on elements of H is the given one, while the coproduct on elements of C is modified to

$$\Delta_{\rtimes} c = c_{(1)} c_{(2)\,[1]} \otimes c_{(2)\,[\underline{2}]}. \tag{6}$$

Here brackets denote the coaction to distinguish it from the coproduct. This is the straightforward equivalent to a semidirect product of groups in quantum group language. For the general theory of crossed products of Hopf algebras see [16].

2 Noncommutative $\mathbb{R}^d$ as a Twist

Part of the discussion in this section reproduces [17]. In particular, the 2-cocycle (10) was found there, and it was shown to give rise to the deformed product (11). However, the full representation theoretic picture essential to our treatment of quantum field theory was lacking. We provide it here.

We work over the complex numbers from now on. Although we use the purely algebraic language for convenience, Hopf algebras are to be understood in a topological sense in the following. Tensor products are appropriate completions. One could use the setting of Hopf C^*-algebras for example [24]. However, our discussion is independent of the functional analytic details and so we leave them out. When referring to function algebras one should have in mind a class compatible with the functional analytic setting chosen.

Consider $\mathbb{R}^d$ as the group of translations of d-dimensional Euclidean space. In the language of quantum groups, the corresponding object is the

Hopf algebra $\mathcal{C}(\mathbb{R}^d)$ of functions on $\mathbb{R}^d$. We can view this as (a certain completion of) the unital commutative algebra generated by the coordinate functions $\{x^1, \ldots, x^d\}$. The product is $(f \cdot g)(x) = f(x) \cdot g(x)$, the unit $1(x) = 1$, the counit $\epsilon(f) = f(0)$, and the antipode $(\mathrm{S}\,f)(x) = f(-x)$. Identifying the (completed) tensor product $\mathcal{C}(\mathbb{R}^d) \otimes \mathcal{C}(\mathbb{R}^d)$ as the functions on the cartesian product $\mathcal{C}(\mathbb{R}^d \times \mathbb{R}^d)$ the coproduct encodes the group law of translation via $\Delta(f)(x, y) = f(x + y)$. We can formally write this as a Taylor expansion

$$\Delta f = \exp\left(x^\mu \otimes \frac{\partial}{\partial x^\mu}\right)(1 \otimes f) = \exp\left(\frac{\partial}{\partial x^\mu} \otimes x^\mu\right)(f \otimes 1).$$

We have the usual $*$-structure $(x^\mu)^* = x^\mu$ making $\mathcal{C}(\mathbb{R}^d)$ into a Hopf $*$-algebra. $\mathcal{C}(\mathbb{R}^d)$ is naturally equipped with the trivial coquasitriangular structure $\mathcal{R} = \epsilon \otimes \epsilon$.

Taking the dual point of view, we consider the Lie algebra of translation generators with basis $\{p_1, \ldots, p_d\}$. We denote its universal envelope by $\mathsf{U}(\mathbb{R}^d)$. Expressing elements of $\mathsf{U}(\mathbb{R}^d)$ as functions in the p_μ, we obtain the same Hopf algebra structure as for $\mathcal{C}(\mathbb{R}^d)$. We define the dual pairing by

$$\langle f(p_\mu), g \rangle = f\left(i\frac{\partial}{\partial x^\mu}\right) g(x)\Big|_{x=0}.$$

The corresponding (left) action of $\mathsf{U}(\mathbb{R}^d)$ on $\mathcal{C}(\mathbb{R}^d)$ that leaves this pairing invariant is given by

$$(p_\mu \triangleright g)(x) = -i\frac{\partial}{\partial x^\mu}g(x).$$

Viewing $\mathsf{U}(\mathbb{R}^d)$ as momentum space, we have the usual translation covariant Fourier transform $\hat{\ } : \mathcal{C}(\mathbb{R}^d) \to \mathsf{U}(\mathbb{R}^d)$ and its inverse given by

$$\hat{f}(p) = \int \frac{\mathrm{d}^d x}{(2\pi)^{d/2}} f(x) e^{-ip_\mu x^\mu} \quad \text{and} \quad f(x) = \int \frac{\mathrm{d}^d p}{(2\pi)^{d/2}} \hat{f}(p) e^{ip_\mu x^\mu}. \quad (7)$$

Now, let θ be a real valued antisymmetric $d \times d$ matrix. Consider the map $\chi_\theta : \mathcal{C}(\mathbb{R}^d) \otimes \mathcal{C}(\mathbb{R}^d) \to \mathbb{C}$ given by

$$\chi_\theta(f \otimes g) = (\epsilon \otimes \epsilon) \circ \exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}\right)(f \otimes g). \quad (8)$$

One easily verifies (check (2) and (3)) that this defines a unital 2-cocycle on $\mathcal{C}(\mathbb{R}^d)$ with inverse $\chi_\theta^{-1} = \chi_{-\theta} = \chi_\theta \circ \tau$ (τ the flip map). Thus, according to section 1 it gives rise to a twisted Hopf algebra $\mathcal{C}_\theta(\mathbb{R}^d)$. However, the twisted product is the same as the original product, i.e., $\mathcal{C}(\mathbb{R}^d)$ and $\mathcal{C}_\theta(\mathbb{R}^d)$ are identical as Hopf algebras. In other words – the group of translations remains unchanged. In fact, it is easy to see from (4) that this must be

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so for any twist on a cocommutative Hopf algebra. The coquasitriangular structure does change on the other hand, and we obtain

$$\mathcal{R}_\theta(f \otimes g) = (\epsilon \otimes \epsilon) \circ \exp\left(-i\,\theta^{\mu\nu}\frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}\right)(f \otimes g).$$

according to (5). In particular, this means that the category of comodules of $\mathcal{C}_\theta(\mathbb{R}^d)$ is equipped with a braiding ψ_θ that is not the flip map. Using (1) we obtain

$$\psi_\theta(f \otimes g) = \exp\left(-i\,\theta^{\mu\nu}\frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}\right)(g \otimes f). \tag{9}$$

In more conventional language this means that the representations of the translation group aquire non-trivial statistics. Note that $\mathcal{R}_\theta^{-1} = \mathcal{R}_\theta \circ \tau$ (with τ the flip map), i.e., $\mathcal{R}_\theta$ is cotriangular (as it must be, being obtained by twisting from a trivial $\mathcal{R}$). Consequently, the braiding is symmetric, i.e., $\psi_\theta^2 = \mathrm{id}$.

By duality we can equivalently express this twist as an invertible element $\Phi_\theta \in \mathsf{U}(\mathbb{R}^d) \otimes \mathsf{U}(\mathbb{R}^d)$ obeying the dual axioms of (2) and (3). We get

$$\Phi_\theta = \exp\left(-\frac{i}{2}\theta^{\mu\nu}p_\mu \otimes p_\nu\right). \tag{10}$$

This is (3.10) in [17]. As in the above discussion the twisted $\mathsf{U}_\theta(\mathbb{R}^d)$ is the same as $\mathsf{U}(\mathbb{R}^d)$ as a Hopf algebra, but the quasitriangular structure becomes nontrivial.

Now consider d-dimensional Euclidean space with an action of the translation group (from the left say). In quantum group language this means that we take a second copy $\widetilde{\mathcal{C}}(\mathbb{R}^d)$ of $\mathcal{C}(\mathbb{R}^d)$ as a left $\mathcal{C}(\mathbb{R}^d)$-comodule algebra. In contrast to the quantum group $\mathcal{C}(\mathbb{R}^d)$ its algebra structure *is* changed under the twist. This is the situation of Example 1.4. Furthermore, we know from Proposition 1.5 that the new product on the twisted $\widetilde{\mathcal{C}}(\mathbb{R}^d)$ which we denote by $\widetilde{\mathcal{C}}_\theta(\mathbb{R}^d)$ is a deformation quantisation. We find

$$(f \star g)(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\xi^\mu}\frac{\partial}{\partial\eta^\nu}\right)f(x+\xi)g(x+\eta)\Big|_{\xi=\eta=0}, \tag{11}$$

which is known as a Moyal $\star$-product [3]. Note that the inherited $*$-structure is compatible with the new algebra structure making $\widetilde{\mathcal{C}}_\theta(\mathbb{R}^d)$ into a $*$-algebra.

According to Theorem 1.2, the category of $\mathcal{C}(\mathbb{R}^d)$-comodules and the category of $\mathcal{C}_\theta(\mathbb{R}^d)$-comodules are equivalent. While objects remain the same under twisting the tensor product does not. In particular, this means that while for $f \in \widetilde{\mathcal{C}}(\mathbb{R}^d)$ the corresponding $f_\theta \in \widetilde{\mathcal{C}}_\theta(\mathbb{R}^d)$ is just the same function this is not so for functions of several variables. In our context a function of n variables is an element of $\widetilde{\mathcal{C}}(\mathbb{R}^d \times \cdots \times \mathbb{R}^d)$ which we write as the tensor

product $\widetilde{\mathcal{C}}(\mathbb{R}^d) \otimes \cdots \otimes \widetilde{\mathcal{C}}(\mathbb{R}^d)$. This is transformed to the tensor product $\widetilde{\mathcal{C}}_\theta(\mathbb{R}^d) \otimes \cdots \otimes \widetilde{\mathcal{C}}_\theta(\mathbb{R}^d)$ via the functor σ_n^{-1}. Explicitly, we obtain

$$f_\theta(x_1, \ldots, x_n) = \exp\left(-\frac{i}{2} \sum_{l<m} \theta^{\mu\nu} \frac{\partial}{\partial x_l^\mu} \frac{\partial}{\partial x_m^\nu}\right) f(x_1, \ldots, x_n). \quad (12)$$

Due to duality (left) $\mathcal{C}(\mathbb{R}^d)$-comodules are really the same thing as (left) $\mathsf{U}(\mathbb{R}^d)$-modules. In particular, viewing momentum space as a left $\mathsf{U}(\mathbb{R}^d)$-module (coalgebra) denoted by $\widetilde{\mathsf{U}}(\mathbb{R}^d)$, it lives in the same category as $\widetilde{\mathcal{C}}(\mathbb{R}^d)$ and we denote its twisted analogue by $\widetilde{\mathsf{U}}_\theta(\mathbb{R}^d)$. The momentum space version of equation (12) reads

$$f_\theta(p^1, \ldots, p^n) = \exp\left(\frac{i}{2} \sum_{l<m} \theta^{\mu\nu} p_\mu^l p_\nu^m\right) f(p^1, \ldots, p^n). \quad (13)$$

The transformation of morphisms (i.e. intertwiners) by $\mathcal{F}$ is non-trivial only if they transform tensor products to tensor products. In particular, this means that integration and Fourier transform (7) are preserved by the twist. Note that even the Fourier transform in several variables survives the twist unchanged, since it factors into Fourier transforms in each variable.

2.1 A Remark on the Noncommutative Torus

All constructions we have made for noncommutative $\mathbb{R}^d$ apply equally to the noncommutative torus. We simply restrict to periodic functions. To be more specific, let $\mathbb{T}^d$ denote the group $U(1)^d$ of translations on the d-dimensional torus of unit radius which we also denote by $\mathbb{T}^d$. The Hopf algebra of functions $\mathcal{C}(\mathbb{T}^d)$ on $\mathbb{T}^d$ has a basis of Fourier modes $\{u_k\}$ for $k \in \mathbb{Z}^d$. We can identify u_k as a periodic function in $\mathcal{C}(\mathbb{R}^d)$ via $u_k(x) = \exp(i k_\mu x^\mu)$. For completeness we provide the relevant formulas explicitly: Product and coproduct are given by $u_k u_l = u_{k+l}$ and $\Delta u_k = u_k \otimes u_k$. The counit is $\epsilon(u_k) = 1$. Antipode and $*$-structure are $\mathrm{S}\, u_k = u_k^* = u_{-k}$. The twist (8) takes the form

$$\chi_\theta(u_k \otimes u_l) = \exp\left(-\frac{i}{2} \theta^{\mu\nu} k_\mu l_\nu\right)$$

and the twisted comodule algebra $\widetilde{\mathcal{C}}_\theta(\mathbb{T}^d)$ satisfies the product rule

$$u_k \star u_l = \exp(-i\, \theta^{\mu\nu} k_\mu l_\nu)\, u_l \star u_k.$$

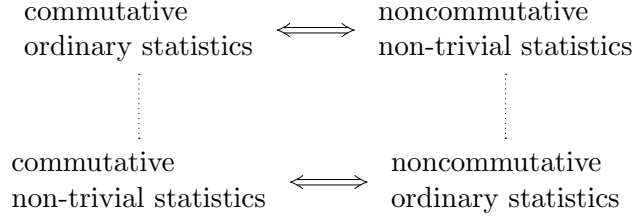|	
commutative ordinary statistics	noncommutative non-trivial statistics
commutative non-trivial statistics	noncommutative ordinary statistics

Figure 1: Relations between quantum field theories on $\mathbb{R}^d$. The arrows indicate equivalences while the dotted lines indicate equality of planar Feynman diagrams.

3 Quantum Field Theory on Noncommutative $\mathbb{R}^d$

Let us examine the noncommutative $\mathbb{R}^d$ with a view towards taking it as the space-time of a quantum field theory. Recall that the coordinate functions $x^1, \ldots, x^d$ of noncommutative $\mathbb{R}^d$ obey commutation relations of the form

$$[x^\mu, x^\nu] = i\, \theta^{\mu\nu} \tag{14}$$

for θ a real valued antisymmetric $d \times d$ matrix. More precisely, noncommutative $\mathbb{R}^d$ is a deformation quantisation of the algebra of functions on ordinary $\mathbb{R}^d$ satisfying (11).

Apart from space-time itself, its group of isometries plays a fundamental role in quantum field theory. After all, fields and particles are representations of this group (or its universal cover) and it leaves a quantum field theory as a whole (i.e., its n-point functions) invariant. What is this group for noncommutative $\mathbb{R}^d$? For general θ, the commutation relations (14) are clearly not invariant under rotations or boosts. However, they are invariant under ordinary translations $x^\mu \mapsto x^\mu + a^\mu$. Thus, it appears natural to let the translations play the role of isometries of noncommutative $\mathbb{R}^d$. This is an important ingredient for the following discussion. We later come back to the question of a possible larger group of symmetries.

It was shown in Section 2 how ordinary $\mathbb{R}^d$ is turned into noncommutative $\mathbb{R}^d$ by a process of twisting. This is induced by a 2-cocycle χ_θ on the quantum group $\mathcal{C}(\mathbb{R}^d)$ of translations. ("Cocycle" here has the meaning dual to that of ordinary group cohomology.) At the same time $\mathcal{C}(\mathbb{R}^d)$ is turned into the quantum group $\mathcal{C}_\theta(\mathbb{R}^d)$. While this still corresponds to the ordinary group of translations, it is different from $\mathcal{C}(\mathbb{R}^d)$ as a quantum group. The difference is encoded in the coquasitriangular structure $\mathcal{R}_\theta$ which is now non-trivial. It equips noncommutative $\mathbb{R}^d$ with a non-trivial statistics encoded in the braiding ψ_θ. This twist-transformation is represented in Figure 1 by the upper arrow. It goes both ways since we can undo the twist by using the inverse 2-cocycle $\chi_{-\theta}$.
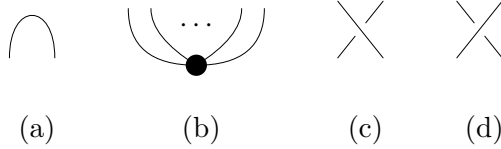
(a)   (b)   (c)   (d)

Figure 2: Free propagator (a) and vertex (b). Over-crossing (c) and under-crossing (d).

What about noncommutative $\mathbb{R}^d$ with *ordinary* statistics? After all, this is the space which has been of interest in the literature. Untwisting this space yields the commutative $\mathbb{R}^d$ as before. However, as before, twisting also exchanges ordinary with braided statistics. Only this time the other way round: We obtain commutative $\mathbb{R}^d$ equipped with braided statistics. This is represented by the lower arrow in Figure 1. In the language of Section 2, we consider $\widetilde{\mathcal{C}}_\theta(\mathbb{R}^d)$ (noncommutative $\mathbb{R}^d$) as a comodule of $\mathcal{C}(\mathbb{R}^d)$ (the translation group with ordinary statistics) and apply the twist with the inverse 2-cocycle $\chi_{-\theta}$. We get $\widetilde{\mathcal{C}}(\mathbb{R}^d)$ (ordinary $\mathbb{R}^d$) but as a comodule of $\mathcal{C}_{-\theta}(\mathbb{R}^d)$ (the translation group with braided statistics). The braiding this time is given by $\psi_{-\theta}$ since we have used the inverse twist. Note that the braiding is in both cases symmetric, i.e., ψ^2 is the identity.

We show in the following how the twist equivalence between the respective spaces gives rise to an equivalence of quantum field theories on those spaces. In order to do that we need to express (perturbative) quantum field theory in a purely algebraic language. Also, we need to be able to deal with quantum field theory on spaces carrying a braided statistics. Both is accomplished by braided quantum field theory [2]. This is briefly reviewed in Section 3.1 and specialised for the case of a symmetric braiding. We can then go on to show the equivalences in Section 3.2 and look in more detail at the perturbative consequences in Section 3.3. Finally, in Section 3.4, we turn to the question of what happens with additional symmetries under twist.

Note that while the whole discussion is solely in terms of $\mathbb{R}^d$ for convenience, everything applies equally to the torus. This follows from the remarks in Section 2.1. The only exception are the extra space-time symmetries considered in Section 3.4.1.

3.1   Symmetric Braided Quantum Field Theory

Braided quantum field theory employs the same path integral formulation and perturbation expansion as ordinary quantum field theory. However, one has to be much more careful since ordering is relevant even inside the path integral due to the braided statistics of the underlying space. As a consequence, one needs to impose extra restrictions to the way Feynman diagrams are drawn. It was shown in [2] how this is accomplished by using
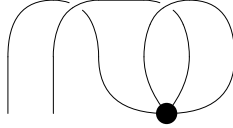
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Figure 3: One of the tadpole diagrams in braided quantum field theory.

an adapted version of the diagrammatic language for braided categories. Let us briefly recall the rules: Diagrams are drawn such that all external legs end on the bottom line. Vertices are drawn with all legs pointing upwards (see Figure 2.b). Free propagators are arches (Figure 2.a) connecting vertices and/or external legs. Over- and under-crossings of lines (Figure 2.c&d) are distinct. Diagrams are evaluated from top to bottom. Horizontally parallel strands correspond to tensor products with each strand representing a field. Denoting the space of fields by X, a free propagator is an element of $X \otimes X$. A vertex with n legs is a map $X \otimes \cdots \otimes X \to \Bbbk$ with the tensor product being n-fold. Over- and under-crossings correspond to the braiding map $\psi : X \otimes X \to X \otimes X$ and its inverse, encoding the non-trivial statistics. The diagram as a whole is an element of $X \otimes \cdots \otimes X$ with as many factors as external lines. Notice that propagators, vertices, and diagrams are by construction invariant under the given (quantum) group of space-time symmetries. As an example, Figure 3 shows a diagram contributing to the 2-point function at 1-loop order in ϕ^4-theory. It corresponds to a tadpole diagram (Figure 4.b) in ordinary quantum field theory.

In the case of a symmetric braiding, over- and under-crossings become the same and the situation is simplified considerably. The complication in the general braided case really is that a permutation of components in a tensor product depends on what transpositions (using ψ or ψ^{-1}) have occured and in which order. This defines a representation of the braid group. In the symmetric case however, any sequence of transpositions (using $\psi = \psi^{-1}$) leading to a given permutation is equivalent. Thus, we have a representation of the symmetric group. For the Feynman diagrams this means that we can return to the usual freedom of ordinary quantum field theory in drawing them. Any way of rearranging an ordinary Feynman graph so that it conforms with the stricter rules of general braided Feynman graphs is equivalent due to the symmetry of the braiding. This supposes that the free propagator is invariant under the braiding ψ. (This is satisfied in the concrete cases considered.)

An analogous simplification occurs with respect to the perturbation expansion. In ordinary quantum field theory the combinatorics of which diagrams are generated is encoded in Wick's theorem. In braided quantum field theory the corresponding role is played by the braided generalisation of Wick's theorem [2]. In the symmetric case, this again reduces to the

ordinary Wick's theorem.

While in the general braided case a diagram is evaluated strictly from top to bottom, this can be relaxed to the ordinary way of evaluating a diagram for the symmetric case. With one crucial exception: Every crossing of lines in a diagram is associated with the braiding ψ. If ψ is not just the flip map, we obtain an extra Feynman rule for crossings. (For the commutative $\mathbb{R}^d$ with braided statistics this is e.g. the rule depicted in Figure 5.)

3.2 The Equivalences for Quantum Field Theory

With the machinery of symmetric braided quantum field theory in place we can handle quantum field theory on any of the versions of $\mathbb{R}^d$ represented in Figure 1. Recall that the arrows in this figure represent twist transformations between the respective spaces. Now, by Theorem 1.2 the twist induces an equivalence between the whole categories of translation covariant objects and maps in which those spaces live. But, as demonstrated in the previous section, the whole perturbation expansion takes place in this category, including Feynman diagrams and n-point functions. This is made explicit by using braided Feynman diagrams and associating the space of fields, tensor products and intertwining maps (vertices, the braiding etc.) with elements of those diagrams. Consequently, quantum field theories on spaces related by twist are equivalent. In particular, the arrows in Figure 1 stand for such equivalences. For an n-point function, a Feynman diagram or a vertex the relation between the commutative quantity G and the noncommutative quantity G_{NC} is in both cases given in momentum space by

$$G_{NC}(p^1, \ldots, p^n) = \exp\left(\frac{i}{2} \sum_{l<m} \theta^{\mu\nu} p_\mu^l p_\nu^m \right) G(p^1, \ldots, p^n) \qquad (15)$$

which is just formula (13) from Section 2. The corresponding position space version is (12).

We would like to stress that our treatment applies to fields in any representation of the translation group and thus to quantum field theory in general. For scalars the space of fields is simply $\widetilde{\mathcal{C}}_\theta(\mathbb{R}^d)$ itself. Any other field lives in a bundle associated with the frame bundle (or its universal cover – the spin bundle) which in particular carries an action of the translation group. Choosing a trivialisation "along translation" allows to write the space of sections of the bundle as $V \otimes \widetilde{\mathcal{C}}(\mathbb{R}^d)$ with translations acting trivially on V. Thus, under twist we obtain $V \otimes \widetilde{\mathcal{C}}_\theta(\mathbb{R}^d)$ with the V-component not being affected at all by the twist. In other words: Extra indices like spinor or tensor indices just show the ordinary behaviour and can be considered completely separate from the noncommutativity going on in space-time. This also applies to the anticommutation of fermions.

Let us make an extra remark about gauge theories. For a gauge bundle there is no canonical action of the translation group. Choosing such an
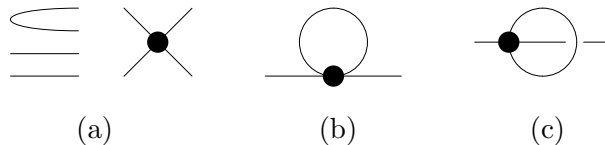
Figure 4: Building blocks for the diagrams of the first order contribution to the 2-point function in ϕ^4-theory (a). Resulting tadpole diagram (b). In the cyclic case diagram (c) is non-equivalent.
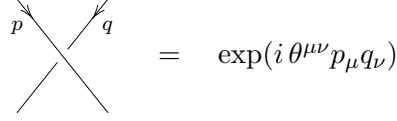
action is the same thing as choosing a trivialisation, i.e., a "preferred gauge". Given such a choice we can treat gauge theory with the above methods. This supposes that we have integrated out the gauge degrees of freedom in the path integral in the usual way, say by the Faddeev-Popov method.

Rigorously speaking, our treatment so far has assumed that quantities encountered in the calculation of Feynman diagrams are finite. Then the transformation (15) between quantum field theories connected by arrows in Figure 1 is straightforward. In order to establish the equivalence not only for finite but also for renormalisable quantum field theories, we need to extend the twisting equivalence to the regularisation process involved in the renormalisation. The only condition for the twist transformation to work in this context is that we remain in the translation covariant category, i.e., that the regularisation preserves covariance under translations. This is easily accomplished. For example, a simple momentum-cutoff regularisation would do, or a Pauli-Villars regularisation. (Note however, that the popular dimensional regularisation can not be used here.) Using such a regularisation, the twisting equivalence holds at every step of the renormalisation procedure, in particular for the renormalised quantities at the end.

As a further remark, the equivalences should also hold non-perturbatively, since the n-point functions (perturbative or not) naturally live in the respective categories. However, for lack of a general non-perturbative method, we can obviously not demonstrate this explicitly. Turning the argument round, one could say that for a well defined theory on one side the transformation (15) *defines* the respective equivalent theory.

3.3   Perturbative Consequences

Let us explore the consequences of the equivalences in terms of perturbation theory. We first discuss the issue of vertex symmetry. It has been observed that vertices which are totally symmetric under an exchange of legs retain only a cyclic symmetry on noncommutative $\mathbb{R}^d$ (with ordinary statistics). Following the upper arrow in Figure 1 from left to right we retain total

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$$\exp(i\,\theta^{\mu\nu}p_\mu q_\nu)$$

Figure 5: Extra Feynman rule for crossings with braided statistics. The arrows indicate the direction of the momenta p and q.

symmetry. For a transposition this takes the form

$$G_{NC}(p^1,\ldots,p^i,p^{i+1},\ldots,p^n)$$
$$= \exp\left(-i\,\theta^{\mu\nu}p_\mu^i p_\nu^{i+1}\right)G_{NC}(p^1,\ldots,p^{i+1},p^i,\ldots,p^n).$$

However, "stripping off" the non-trivial braiding, i.e., considering ordinary transpositions by flip, leaves only a cyclic symmetry. Following the lower arrow to the left, we have the opposite situation. Vertices are now ordinarily totally symmetric, but we have a non-trivial braiding with respect to which they are only cyclic symmetric.

The deeper reason for the retention of cyclic symmetry is a property of the coquasitriangular structure $\mathcal{R}_\theta$ defining the braiding. As a consequence of this property, cyclic symmetry with respect to ordinary and the braided statistics is the same for translation invariant objects like vertices. This is Lemma 1.1. For perturbation theory the use of vertices that are only cyclic symmetric means that diagrams which would be the same for total symmetry may now differ. Consider for illustration the 2-point function in ϕ^4-theory at 1-loop order. Assembling the building blocks (Figure 4.a) in all possible ways (noting that the legs of the propagator are to be considered identical) results in 8 times diagram (b) plus 4 times diagram (c), given only cyclic symmetry of the vertex (see Figure 4). A total symmetry would imply that both diagrams are equal, leading to the usual factor of 12.

Now, recall from Section 3.1 that a non-trivial braided statistics leads to the appearance of an extra Feynman rule. The braiding map ψ instead of the trivial exchange is now associated with each crossing. In fact, this is the only effect of the (symmetric) braiding in perturbation theory. It follows immediately that planar Feynman diagrams are identical in theories that differ only by their (symmetric) braided statistics. This is indicated by the dotted lines in Figure 1.

Filk's result [11] for planar diagrams is an immediate consequence: We evaluate a planar diagram in the commutative setting and follow the lower arrow in Figure 1 to the right. Diagrams are simply related by the equivalence formula (15). For non-planar diagrams we also use the commutative setting. We only have to take into account the crossing factors from the non-trivial statistics. They are given by the extra Feynman rule in Figure 5. This is the momentum space version of formula (9) with opposite sign for θ.

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If we aggregate the factors for a given diagram by encoding all the crossings into an intersection matrix, we obtain an overall factor

$$\exp\left(i\sum_{k>l} I_{kl}\theta^{\mu\nu}p^k_\mu p^l_\nu\right). \tag{16}$$

Here, the indices k,l run over all lines of the diagram and I_{kl} counts the oriented number of intersections between lines k and l. Then again, relation (15) leads to the noncommutative theory. This is Filk's result for non-planar diagrams [11]. Note that it was already observed in [15] that (16) can be obtained by assigning phase factors to crossings.

As a further remark, it has been observed that quadratic terms in the Lagrangian are not modified in the noncommutative setting. This follows from a property of the twisting cocycle. Any invariant object with 2 components (like a 2-leg vertex, a free propagator etc.) remains unchanged by the twist. This is Lemma 1.3.

3.4 Additional Symmetries

In this final section we consider the effect of twisting on additional symmetries. We follow the upper arrow in Figure 1 from left to right.

3.4.1 Space-Time Symmetry

As mentioned before, the commutation relations (14) are not invariant under rotations. However, ordinary Euclidean space is, and since the noncommutative version is a twist of the commutative one, there should be an analogue of those symmetries. This is indeed the case. Consider the group of (orientation preserving) rotations $SO(d)$ in d dimensions. In quantum group language we consider the algebra of functions $\mathcal{C}(SO(d))$ generated by the matrix elements t^μ_ν of the fundamental representation. We have relations $t^\mu_\rho t^\nu_\rho = \delta^\mu_\nu = t^\rho_\mu t^\rho_\nu$ (summation over ρ implied) and $\det(t) = 1$, coproduct $\Delta\, t^\mu_\nu = t^\mu_\rho \otimes t^\rho_\nu$, counit $\epsilon(t^\mu_\nu) = \delta^\mu_\nu$, and antipode $S\, t^\mu_\nu = t^\nu_\mu$. We have a $*$-structure given by $(t^\mu_\nu)^* = t^\nu_\mu$.

We extend the translation group $\mathbb{R}^d$ to the full group $E := \mathbb{R}^d \rtimes SO(d)$ of (orientation preserving) isometries of Euclidean space. I.e. we consider the Hopf algebra $\mathcal{C}(E) = \mathcal{C}(\mathbb{R}^d \rtimes SO(d)) \cong \mathcal{C}(\mathbb{R}^d) \rtimes \mathcal{C}(SO(d))$. The rotations (co)act on the translations from the left by $x^\mu \mapsto t^\mu_\nu \otimes x^\nu$. The resulting semidirect product Hopf algebra is generated by x^μ and t^μ_ν with the given relations. The coproduct of t^μ_ν remains the same but for x^μ we now obtain $\Delta\, x^\mu = x^\mu \otimes 1 + t^\mu_\nu \otimes x^\nu$. (Use (6).) This also determines the left coaction on $\widetilde{\mathcal{C}}(\mathbb{R}^d)$.

The cocycle χ_θ on $\mathcal{C}(\mathbb{R}^d)$ extends trivially to a cocycle on the larger quantum group $\mathcal{C}(E)$, i.e., we let χ_θ just be the counit on the generators of

$\mathcal{C}(SO(d))$. The twist *does* change the algebra structure now. This was to be expected since we have already seen that ordinary rotation invariance is lost. What do we have instead? Using (4) and (8) we find that the relations for the x^μ become

$$x^\mu \bullet x^\nu - x^\nu \bullet x^\mu = i\,\theta^{\mu\nu} - i\,\theta^{\rho\sigma}t^\mu_\rho \bullet t^\nu_\sigma,$$

while the t^μ_ν still commute with the other generators. Thus, the twisted space-time symmetries $\mathcal{C}_\theta(E)$ form a genuine quantum group (noncommutative Hopf algebra), no longer corresponding to any ordinary group.

When dealing with the translation group alone, we were able to remove the non-trivial coquasitriangular structure responsible for the braided statistics and replace it by a trivial one (follow the dotted line on the right in Figure 1 downwards). However, this is no longer possible for the whole Euclidean motion group. A genuine quantum group as the one obtained here does not admit a trivial coquasitriangular structure. Thus, removing the braided statistics really breaks the symmetry for the quantum field theory. (Note that the argument applies to Minkowski space and the Poincaré group in the identical way.)

3.4.2 Gauge Symmetry

Let us consider a gauge theory with gauge group G. The gauge transformations are the maps $\mathbb{R}^d \to G$. We denote the group of such maps by $\Gamma = \{\mathbb{R}^d \to G\}$. The symmetry group generated by translations and gauge transformations is the semidirect product $\Omega := \Gamma \rtimes \mathbb{R}^d$, where we have chosen an action of the translation group on the gauge bundle. (Note that the inclusion of further space-time symmetries does not modify the argument.) While the group Ω "forgets" about the trivialisation of the gauge bundle corresponding to the chosen action, we do need the trivialisation to extend the twisting cocycle from $\mathbb{R}^d$ to Ω. This is in accordance with our remark on gauge theories above. In quantum group language we have the semidirect product of Hopf algebras $\mathcal{C}(\Omega) = \mathcal{C}(\Gamma \rtimes \mathbb{R}^d) \cong \mathcal{C}(\Gamma) \rtimes \mathcal{C}(\mathbb{R}^d)$. The cocycle χ_θ extends trivially from $\mathcal{C}(\mathbb{R}^d)$ to $\mathcal{C}(\Gamma)$. Applying the twist (4) with (6) results in a noncommutative product

$$f \bullet \gamma = \chi_\theta(f_{(1)} \otimes \gamma_{(1)})\, f_{(2)}\gamma_{(2)},$$
$$\gamma \bullet \omega = \chi_\theta(\gamma_{(1)} \otimes \omega_{(1)})\, \gamma_{(2)}\omega_{(2)},$$

while $f \bullet g = fg$ for $f, g \in \mathcal{C}(\mathbb{R}^d)$ and $\gamma, \omega \in \mathcal{C}(\Gamma)$. Thus, the group of gauge transformation does not survive the twist as an ordinary group. As for the case of rotations we find that we obtain a genuine quantum group. Again, the removal of the braided statistics would break the symmetry. Note that this does not exclude the possibility of different gauge symmetries. See [25] and more recently [13] for discussions of gauge theory on noncommutative $\mathbb{R}^d$.

Acknowledgements

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